Fokker-Planck equation for Boltzmann-type and active particles: Transfer probability approach

S. A. Trigger*
Joint Institute for High Temperatures, Russian Academy of Sciences, 13/19 Izhorskaya Street, Moscow 127412, Russia
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A Fokker-Planck equation with velocity-dependent coefficients is considered for various isotropic systems on the basis of probability transition (PT) approach. This method provides a self-consistent and universal description of friction and diffusion for Brownian particles. Renormalization of the friction coefficient is shown to occur for two-dimensional and three-dimensional cases, due to the tensorial character of diffusion. The specific forms of PT are calculated for Boltzmann-type and absorption-type collisions (the latter are typical in dusty plasmas and some other systems). The validity of the Einstein's relation for Boltzmann-type collisions is analyzed for the velocity-dependent friction and diffusion coefficients. For Boltzmann-type collisions in the region of very high grain velocity as well as it is always for non-Boltzmann collisions, such as, absorption collisions, the Einstein relation is violated, although some other relations (determined by the structure of PT) can exist. The generalized friction force is investigated in dusty plasmas in the framework of the PT approach. The relation among this force, the negative collecting friction force, and scattering and collecting drag forces is established. The concept of probability transition is used to describe motion of active particles in an ambient medium. On basis of the physical arguments, the PT for a simple model of the active particle is constructed and the coefficients of the relevant Fokker-Planck equation are found. The stationary solution of this equation is typical for the simplest self-organized molecular machines.

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I. INTRODUCTION

Brownian dynamics nowadays is in the focus of interest due to the wide new fields of applications: physical-chemical systems, so-called active walkers, e.g., cells and other objects in biological systems, dusty plasmas with natural and artificial grains and many other systems. The characteristic property of such systems is velocity-dependent friction and diffusion coefficients. The existence of the Einstein relation and even the correct specific forms of the Fokker-Planck equation for such systems are not still completely clarified. In particular, attempts to use the Langevin equation as a stochastic basis for derivation of the Fokker-Planck equation lead to non-sign-valued result. The different forms of the Fokker-Planck equation, such as so-called Ito and Stratonovich [1–4] ones, appear. For the systems close to equilibrium, Brownian particles keep stationary random motion under action of the stochastic forces, which are compensated by the particle friction and thus, the work produced by the Langevin sources is equal to the energy dissipated in course of the Brownian particle motion. This energy balance is described by the fluctuation-dissipation theorem in the form of the Einstein law. Obviously, the fluctuation-dissipation theorem and the Einstein relation can be violated in the case of nonequilibrium systems (even in the stationary case), in particular, in the open systems. Starting from the classical Lord Rayleigh work [5] many studies of the nonequilibrium motion of Brownian particles with an additional (inner or external) energy supply have been performed. In particular, such studies are of great importance in physical-chemical [6,7] and biological [8] systems, in which nonequilibrium Brownian particle motion is referred to as the motion of active Brownian particles. The dynamical and energetic aspects of motion for the active Brownian particles have been described recently on the basis of the Langevin equation and the appropriate Fokker-Planck equation [9,10]. Possibility of negative friction (negative values of the friction coefficient) for Brownian particles was regarded as a result of energy pumping. For some phenomenological dependence of the friction coefficient as a function of the grain's velocity, a one-particle stationary non-Maxwellian distribution function was found.

The traditional formulations of the nonequilibrium Brownian motion are based on some phenomenological expressions for the friction and diffusion coefficients. In particular, it means that deviations from the Einstein relation as well as the velocity dependence of these coefficients are postulated and a high level of uncertainty for application of such models to the real systems takes place. Recently we considered another situation, when the kinetic coefficients can be calculated explicitly on the basis of microscopically derived Fokker-Planck equation for dusty plasmas [11,12]. It was recently shown [13] that in the case of strong Coulomb interaction of highly charged grains in dusty plasmas, due to ion absorption by grains, the friction coefficient can become negative. The necessary criterion for negative friction due to ion absorption is found as $\Gamma \equiv e^2 Z_g Z_i / a T_i > 1$ (here Z_g, Z_i are the charge numbers for the grains and ions, respectively, a is the grain radius, T_i is the ion temperature). The appropriate threshold value of the grain charge was determined. The stationary solution of the Fokker-Planck equation with the velocity-dependable kinetic coefficients was obtained and the considerable deviation of such solution from the Maxwellian distribution was demonstrated. The physical reason for manifestation of negative friction in that case is clear: the cross section for ion absorption by grain increases, when the relative velocity between the ion and grain decreases, due

*Present address: Humboldt University, 110 Invalidenstrasse, D-10115 Berlin, Germany. Email address: strig@gmx.net

to the charge-dependent part of the cross section. Therefore, for a moving highly charged grain ($\Gamma \gg 1$), the momentum transfer from ions to the grain in the direction of grain velocity can be higher than in the opposite direction.

In this paper we develop more general approach, based on probability transition, to simplify the Fokker-Planck equation and to calculate the velocity-dependent friction and diffusion coefficients for the different systems. On that way we find various forms of probability transition (PT) for Boltzmann-type and for absorption collision integrals. The crucial peculiarity of the exact expressions for the mentioned coefficients follows from the exact representation of these coefficients through the function of probability transition: it is impossible to define the coefficients independently not only for the processes, which describe the systems close to thermodynamic equilibrium when the Einstein relation is *a priori* valid, but also for the systems in which there is a stationary but a nonequilibrium state, or for the systems far from equilibrium; any rigorous approximate model of the Fokker-Planck equation has to be based on self-consistent expressions for the friction and diffusion coefficients, based on the PT.

As an example we consider a wide class of open or far from equilibrium systems, where the Einstein relation is not applicable. For active particles, the suggested consideration can be easily applied by construction of the probability transition on basis of the physical arguments.

II. PROBABILITY TRANSITION AND VELOCITY-DEPENDABLE FRICTION AND DIFFUSION COEFFICIENTS

The appropriate kinetic equation describing motion of Brownian particles in some medium with the momentum exchange may be written as

$$\frac{df_g(\mathbf{P},t)}{dt} = I_g(\mathbf{P},t) = \int d\mathbf{q}\{w(\mathbf{P}+\mathbf{q},\mathbf{q})f_g(\mathbf{P}+\mathbf{q},t) - w(\mathbf{P},\mathbf{q})f_g(\mathbf{P},t)\}, \tag{1}$$

where $f_g(\mathbf{P})$ is the distribution function of Brownian particles (grains) of the mass M. The elementary process is the change of momentum of the grain $\mathbf{P}$ to $(\mathbf{P}-\mathbf{q})$. The probability transition $w(\mathbf{P},\mathbf{q})$ in Eq. (1) describes the probability for grain with linear momentum $\mathbf{P}$ to lose the momentum $\mathbf{q}$. Equation (1) has a form of master equation. In general, the probability transition is the function of time itself. To simplify Eq. (1) for the processes with the momentum transfer $\mathbf{q} \ll \mathbf{P}$, we have expanded the right side of Eq. (1) by $\mathbf{q}$. The result of the expansion is the Fokker-Planck equation for grains:

$$\frac{df_g(\mathbf{P},t)}{dt} = \frac{\partial}{\partial P_i}\left[A_i(\mathbf{P})f_g(\mathbf{P}) + \frac{\partial}{\partial P_j}[B_{ij}(\mathbf{P})f_g(\mathbf{P})]\right]. \tag{2}$$

The coefficients $A_i(\mathbf{P})$ and $B_{ij}(\mathbf{P})$, as easy to see by expansion of Eq. (1), are expressed explicitly through the probability transition $w(\mathbf{P},\mathbf{q})$ by the relations (e.g., Refs. [3,14])

$$A_i(\mathbf{P}) = \int d\mathbf{q} q_i w(\mathbf{P},\mathbf{q}), \tag{3}$$

$$B_{ij}(\mathbf{P}) = \frac{1}{2}\int d\mathbf{q} q_i q_j w(\mathbf{P},\mathbf{q}). \tag{4}$$

Let us assume that probability transition $w(\mathbf{P},\mathbf{q})$ is a function of only two vectors $\mathbf{P}$ and $\mathbf{q}$. It means, for example, that there is no, let us say, drift velocity of the media surrounding the grain, as well as some inner vector inside the grain, which can influence the probability transition. For that case, the general structure of the coefficients A_i and B_{ij} is evident:

$$A_i(\mathbf{P}) = P_i \beta(P),$$

$$B_{ij}(\mathbf{P}) = \frac{P_i P_j}{P^2} B_{\parallel}(P) + \left(\delta_{ij} - \frac{P_i P_j}{P^2}\right) B_{\perp}(P), \tag{5}$$

where $\beta(P)$, $B_{\parallel}(P)$, and $B_{\perp}(P)$ are the functions of modulus P. Let us consider at first the stationary case to understand the form of the Fokker-Planck equation and solution, when the friction and diffusion coefficients are the functions of grain's velocity. On this basis, in particular, the well known problem related to the Ito [1] and Stratonovich [2] forms of Langevin and Fokker-Planck equations [3,4] can be solved and applicability of the Einstein relation for the various kinds of functions of probability transitions can be investigated. The results can be used also for unstationary case, if the initial distribution function is isotropic (which is not always valid, naturally). Then, taking into account isotropy of the distribution function $f_g(\mathbf{P},t)$, the Fokker-Planck equation is given by

$$\frac{df_g(V)}{dt} = \frac{\partial}{\partial V_i}\left[\beta^*(V) V_i f_g(V) + \frac{V_i}{V}\frac{\partial}{\partial V}[D_{\parallel}(V) f_g(V)]\right], \tag{6}$$

or in the equivalent form

$$\frac{df_g(V)}{dt} = \left(s + V\frac{\partial}{\partial V}\right)\left[\beta^*(V) f_g(V) + \frac{1}{V}\frac{\partial}{\partial V}[D_{\parallel}(V) f_g(V)]\right]. \tag{7}$$

Here s is the dimension of the velocity space and the scalar functions of V are the same as ones of P, but expressed via the equality $P = MV$. We use above the velocity variable for grains $\mathbf{V}$ instead of momentum $\mathbf{P}$ and the diffusion tensor $D_{ij}(\mathbf{V}) = M^{-2} B_{ij}(\mathbf{V})$. We also use these notations below. The functions $\beta^*(V)$ and $D_{\parallel}(V)$ are determined via the transition probability as

$$\beta^*(V) = \beta(V) + \frac{s-1}{V^2}[D_{\parallel}(V) - D_{\perp}(V)], \tag{8}$$

$$\beta(V) = \frac{1}{P^2}\int d^s\mathbf{q}(\mathbf{P}\cdot\mathbf{q}) w(\mathbf{P},\mathbf{q}), \tag{9}$$

$$D_{\parallel}(V)=\frac{1}{2M^2P^2}\int d^s\mathbf{q}(\mathbf{P}\cdot\mathbf{q})^2w(\mathbf{P},\mathbf{q}), \tag{10}$$

$$D_{\perp}(V)=\frac{1}{2(s-1)M^2P^2}\int d^s\mathbf{q}[P^2q^2-(\mathbf{P}\cdot\mathbf{q})^2]w(\mathbf{P},\mathbf{q}), \tag{11}$$

where $(\mathbf{P}\cdot\mathbf{q})$ is the scalar product in the velocity space with dimension s. Equation (8) can be rewritten in the form

$$\beta^*(V)=\beta(V)+\frac{1}{2P^2}\int d\mathbf{q}w(\mathbf{P},\mathbf{q})[s(\mathbf{P}\cdot\mathbf{q})^2/P^2-q^2]. \tag{12}$$

We see that the three scalar functions of P, determined by the different moments of probability transition, permit to find the coefficients in the Fokker-Planck equation.

For the anisotropic velocity distribution function or for presence of the external fields, which do not change the friction and diffusion (which is not always valid, naturally), Eq. (2) can be rewritten as

$$\frac{df_g(\mathbf{V})}{dt}=\frac{\partial}{\partial V_i}\left[V_i\beta^*(V)f_g(\mathbf{V})+\frac{V_iV_j}{V^2}\frac{\partial}{\partial V_j}[D_{\parallel}(V)f_g(\mathbf{V})]+\left(\delta_{ij}-\frac{V_iV_j}{V^2}\right)\frac{\partial}{\partial V_j}[D_{\perp}(V)f_g(\mathbf{V})]\right]. \tag{13}$$

For simplicity, external fields are not included in Eq. (13). The useful equivalent representation of Eq. (13) has a form

$$\frac{df_g(\mathbf{V})}{dt}=\frac{\partial}{\partial V_i}\left[V_i\left(\beta^*(V)+\frac{1}{V}\frac{\partial D_{\parallel}(V)}{\partial V}\right)f_g(\mathbf{V})+D_{\parallel}(V)\frac{V_iV_j}{V^2}\frac{\partial f_g(\mathbf{V})}{\partial V_j}+D_{\perp}(V)\left(\delta_{ij}-\frac{V_iV_j}{V^2}\right)\frac{\partial f_g(\mathbf{V})}{\partial V_j}\right]. \tag{14}$$

The stationary solution of the Fokker-Planck equation with the kinetic coefficients from Eqs. (8) and (10) for the grain distribution function $f_g(V)$ is

$$f_g(V)=\frac{C}{D_{\parallel}(V)}\exp\left[-\int_0^V dv v\frac{\beta^*(v)}{D_{\parallel}(v)}\right], \tag{15}$$

where C is a constant, providing normalization. As it is easy to see from Eq. (6) for isotropic case, the stationary (as well as nonstationary) Fokker-Planck equation with the velocity-dependent coefficients has a well defined form and the question of ''renormalized'' friction coefficient is solved completely by Eq. (8). The uncertainty in choice of the Fokker-Planck equation in the forms suggested, e.g., in Refs. [1,2,4], was created by attempts to connect the Langevin and the respective Fokker-Planck equations by one-to-one correspondence, starting from the Langevin equation. The real structure of this renormalization, due to tensorial character of the diffusion $D_{ij}(\mathbf{V})$ as follows from Eq. (8), permits to reformulate the problem: what must be the structure of Langevin equation for s-dimensional case to be relevant to the single-valued Fokker-Planck equation, based on the specific probability transition. Because we use, as the basis, transition probability we can establish the validity or violation of the Einstein relation between the friction and diffusion coefficients directly, without usual suggestion of Maxwellian form of the static distribution function for Brownian particles, which is valid for the equilibrium state (when the Einstein relation is fulfilled *a priori*). In particular, the existence of the Einstein (or some different from one) relation between the momentum-dependable coefficients can be investigated. Correspondence between the Fokker-Planck and Langevin equations for s-dimensional case on the basis of PT approach will be considered in detail in a separate publication. Below we find the probability transition and investigate the various cases for the PT and Fokker-Planck equations.

III. BOLTZMANN-TYPE COLLISIONS

Let us consider the Boltzmann collision integral for two species of particle-light component (called below atoms) with the mass m and grains with the mass M $(m \ll M)$, which interact one with another (generalizations can be easily done). To find for such a process the PT function $w_s(\mathbf{P},\mathbf{q})$ it is enough, for example, to transform the part of the Boltzmann collision integral, describing the loss of grains in the phase volume $d\mathbf{P}$ near the point $\mathbf{P}$, to the variables $\mathbf{P}$ and $\mathbf{q}$, where $\mathbf{q}$ is the momentum transferred during the elementary act of collision between atom and grain. Then, comparing the result of transformation with Eq. (1), for three-dimensional (3D) case we find

$$w_s(\mathbf{P},\mathbf{q})=\frac{m}{\mu}\int d\mathbf{o}\frac{q^2}{2\mu|(\mathbf{q}\cdot\mathbf{n})|}\frac{d\sigma}{d\mathbf{o}}\left(\frac{q^2}{2\mu|(\mathbf{q}\cdot\mathbf{n})|},\chi\right)\times f_n\left[\frac{m}{M}\mathbf{P}-\frac{m}{\mu}\left(\frac{q^2\mathbf{n}}{2|(\mathbf{q}\cdot\mathbf{n})|}-\mathbf{q}\right)\right]. \tag{16}$$

Here $d\mathbf{o}=\sin\chi d\chi d\phi$ is the element of the space angle for scattering with the differential cross section $d\sigma$, f_n is the distribution function of atoms, $\mathbf{q}=(\mathbf{p}'-\mathbf{p})=(\mathbf{P}-\mathbf{P}')$, $q\cos\chi=(\mathbf{q}\cdot\mathbf{n})$, vector $\mathbf{n}$ is the unit vector along the velocity $\mathbf{v}_0'$ of atom after collision in the system of center of mass for colliding particles and μ is the reduced mass. The values of the limiting angles for χ above and below are usually $\chi_{min}=0$ and $\chi_{max}=\pi$, except some special situations when the integral over χ diverges, as for example, for the purely Coulomb interaction when the known cutting with $0<\chi_{min}\leqslant\chi_{max}<\pi$ is necessary. Taking into account the relation $q^2/[2|(\mathbf{q}\cdot\mathbf{n})|]=|(\mathbf{v}-\mathbf{V})|$, Eq. (16) can be rewritten in the form more useful for applications:

$$w_s(\mathbf{V},\mathbf{q})=\int d\Omega\int d\mathbf{v}\delta[\mathbf{q}+\mu\mathbf{v}-\mu|\mathbf{v}|\mathbf{n}]\times f_n(\mathbf{v}+\mathbf{V})|\mathbf{v}|\frac{d\sigma}{d\mathbf{o}}(|\mathbf{v}|,\chi). \tag{17}$$

We can use representation (17) to obtain the useful general expressions for the friction and diffusion coefficients:

$$\beta(\mathbf{V})=\frac{\mu}{\mathbf{M}V^2}\int d\mathbf{\Omega}\int d\mathbf{v}[\mathbf{V}\cdot(|\mathbf{v}|\mathbf{n}-\mathbf{v})]\times f_n(\mathbf{v}+\mathbf{V})|\mathbf{v}|\frac{d\sigma}{d\mathbf{o}}(|\mathbf{v}|,\chi), \tag{18}$$

$$D_{\parallel}(\mathbf{V})=\frac{\mu}{\mathbf{M}V^2}\int d\mathbf{\Omega}\int d\mathbf{v}[\mathbf{V}\cdot(|\mathbf{v}|\mathbf{n}-\mathbf{v})]^2\times f_n(\mathbf{v}+\mathbf{V})|\mathbf{v}|\frac{d\sigma}{d\mathbf{o}}(|\mathbf{v}|,\chi). \tag{19}$$

The similar expression can be written for $D_{\perp}$. When the distribution function of atoms f_n has the Maxwellian form with the temperature T, the function $w(\mathbf{V},\mathbf{q})$ can be simplified, taking into account the inequality $\mathbf{q}\ll\mathbf{P}$.

If $|\mathbf{V}|\ll v_T$, where for the thermal velocity of neutral particles (atoms) we use the notation v_T, the distribution function in PT can be expanded and we arrive, with respective accuracy of the order of $\mu/M\simeq m/M$, at the expression for $w_s(\mathbf{V},\mathbf{q})$:

$$w_s(\mathbf{V},\mathbf{q})=-\frac{m}{T}n_n\left(\frac{m}{2\pi T}\right)^{3/2}\int d\mathbf{\Omega}\int d\mathbf{v}\delta[\mathbf{q}+\mu\mathbf{v}-\mu|\mathbf{v}|\mathbf{n}]\exp\left(-\frac{mv^2}{2T}\right)|\mathbf{v}|(\mathbf{v}\cdot\mathbf{V})\frac{d\sigma}{d\mathbf{o}}(|\mathbf{v}|,\chi). \tag{20}$$

If atoms with the density n_n are considered as the point particles and grains have the radius a, it is easy to find from Eqs. (18)–(20) the values of the coefficients β and $D_{\parallel}$:

$$\beta(V)=8\frac{\sqrt{2\pi}}{3}(m/M)a^2n_nv_T, \tag{21}$$

$$D_{\parallel}(V)=8\frac{\sqrt{2\pi}}{3}(m/M^2)a^2n_nTv_T. \tag{22}$$

We see that the Einstein relation is fulfilled:

$$D(V)=M^{-1}T\beta(V). \tag{23}$$

Calculations of β and $D_{\parallel}$ were done independently on the basis of the appropriate PT function.

Let us now consider under general condition $\mathbf{q}\ll\mathbf{P}$ the opposite case $|\mathbf{V}|\gg v_T$ to solve the problem of the validity of the Einstein relation for the velocity-dependable coefficients of the Fokker-Planck equation for arbitrary values of grain velocities [4]. In fact, the answer can be found already for the particular case of atoms, scattering by grains, considered as the hard spheres. Even in this simple case, as it will be shown, the Einstein relation is violated for high grain velocity. In the limit of extremely high grain velocity we can use the simplest approximation for the distribution function $f_n(\mathbf{v}+\mathbf{V})=n_n\delta(\mathbf{v}+\mathbf{V})$ and from Eqs. (11) and (17)–(19) obtain

$$\beta(V)=(2m/3M)\pi a^2n_nV, \tag{24}$$

$$D_{\parallel}(V)=(m/M^2)\pi a^2n_nV^3. \tag{25}$$

For $D_{\perp}$ by similar calculations we find

$$D_{\perp}(V)=\frac{D_{\parallel}(V)}{4}. \tag{26}$$

Therefore, in that limit instead of the Einstein relation we find another relation between velocity dependable $\beta(V)$ and $D_{\parallel}(V)$:

$$D(V)=\frac{2}{3M}mV^2\beta(V). \tag{27}$$

This situation can be classified as far from equilibrium. For considering the case of uncharged spherical grains, the simplest interpolation relation between the friction and diffusion coefficients can be suggested:

$$D(V)=\frac{T}{M}\left(1+\frac{2mV^2}{3T}\right)\beta(V). \tag{28}$$

The similar interpolations as well as the exact velocity-dependent relations for the arbitrary cross sections can be found from Eqs. (18) and (19). As it follows from the relations considered above, $\beta^*(V)\simeq\beta(V)$ for all the cases with accuracy m/M. In general, by using expansions to higher order of the small relation q/P, the expressions for the functions $\beta^*(V)$, $D_{\parallel}(V)$, and $D_{\perp}(V)$ can be calculated and the possibility to neglect the difference between $\beta^*(V)$ and $\beta(V)$ can be established for the considered Boltzmann-type collisions.

The essential influence of velocity dependence on the values of the friction and diffusion coefficients and the violation of the Einstein relation for the Boltzmann-type collisions take place only for the extreme velocities much higher than the thermal velocity of the light particles. For the case of low grain velocity on basis of general representations (18) and (19) for the friction and diffusion coefficients, the applicability of the Einstein relation for arbitrary cross section of scattering can be shown.

IV. ABSORPTION COLLISIONS

Now turn to other type of collisions, namely, to the absorption collisions, which are typical, for example, in dusty plasmas and some other open systems. As it is well known the process of grain charging by absorption of the electrons and ions by grains leads to the stationary (but nonequilibrium) state in plasma discharges. In so-called OML (orbital motion limited) approximation, the electrons and ions approaching the grain at a distance less than the grain radius a are assumed to be absorbed. It is clear that the absorption collisions cannot be described by the Boltzmann-type collision integral. The appropriate correct form of the absorption collision integral has been postulated and applied in Refs. [14,15]. The rigorous kinetic theory of the electron and ion absorption in dusty plasmas, which exists in parallel with the usual processes of electron and ion scattering by grains, has been developed in Refs. [11,12], where also the Fokker-Planck equation for the charged grains was justified. Below we use the simplest form of the Fokker-Planck equation for grains with a fixed charge (distribution by charge assumed narrow, which is often in reality). Our aim here is to find the probability transition function for absorption and to demonstrate efficacy of such an approach. More complicated cases can be considered similarly. We also ignore increase of the grain mass [16,17], assuming that neutral atoms generated in the course of the surface electron-ion recombination escape from the grain surface into a plasma. Naturally this process also changes the momentum balance for the particles, but we will not consider this process in our present model. Then the kinetic equation for grains can be written as

$$\frac{df_g(\mathbf{P},Q,t)}{dt}=I_g(\mathbf{P},Q,t)$$
$$=\sum_\alpha \int d\mathbf{p} f_\alpha(\mathbf{p})\{W_\alpha(\mathbf{p},\mathbf{P}-\mathbf{p},Q)f_g(\mathbf{P}-\mathbf{p},Q)$$
$$-W_\alpha(\mathbf{p},\mathbf{P},Q)f_g(\mathbf{P},Q)\}, \tag{29}$$

where $f_\alpha(\mathbf{p})$ is the distribution function for the electrons ($\alpha=e$) and ions ($\alpha=i$) and $Q=eZ_g$. The elementary process is the absorption of electron or ion with the mass m_α. The probability of absorption is given by

$$W_\alpha(\mathbf{p},\mathbf{P},Q)=\sigma_c\left(\left|\frac{\mathbf{P}}{M}-\frac{\mathbf{p}}{m_\alpha}\right|,Q\right)\left|\frac{\mathbf{P}}{M}-\frac{\mathbf{p}}{m_\alpha}\right|, \tag{30}$$

where $\sigma_c(v,Q)$ with $v\equiv(|\mathbf{P}/M-\mathbf{p}/m|)$ is the cross sections for absorption (or collection) of the light plasma particles by grain in OML theory:

$$\sigma_c(Q,v)=\pi a^2\left(1-\frac{2e_\alpha Q}{m_\alpha v^2 a}\right)\theta\left(1-\frac{2e_\alpha Q}{m_\alpha v^2 a}\right). \tag{31}$$

Below on the basis of Eqs. (29)–(31), two different but related problems are solved. At first, to consider the problem of absorption in simplest form, we accept the next simplification in spirit of Refs. [16,17], namely, we will consider absorption of neutral atoms of one sort. It means we put $Q=0$ and instead of summation by α retain only notation with the index n (for the neutral atoms) in Eqs. (29)–(31). Generalization of this simplest model to the case of charge absorption with a fixed charge is quite simple.

The second problem in focus of our interest connected with a real dusty plasma, when there is ion stream and so-called drag force, applied to the grains and created by ion absorption and ion scattering exists.

Let us start with a system of neutral particles absorbed by grains. The momentum transferred to the grain due to absorption is equal to the momentum of the atom $\mathbf{p}$ colliding with the grain. Therefore the probability transition $w_c(\mathbf{P},\mathbf{q})$ for the considered case can be immediately found by comparison of Eqs. (1) and (29),

$$w_c(\mathbf{P},\mathbf{q})=f_n(-\mathbf{q})\sigma_c\left(\left|\frac{\mathbf{P}}{M}+\frac{\mathbf{q}}{m}\right|\right)\left|\frac{\mathbf{P}}{M}+\frac{\mathbf{q}}{m}\right|. \tag{32}$$

If we choose the Maxwellian distribution for atoms and assume, to consider the simplest case, that absorption is purely geometrical, $\sigma_c=\pi a^2$ [the particular case of Eq. (31) for $Q=0$], we easily obtain with accuracy $\sim m/M$ that $\beta^*(V)\simeq\beta(V)$. A simple calculation leads to the relation between $\beta(V)$ and $D_\parallel(V)$:

$$D_\parallel(V)=2M^{-1}T\beta(V). \tag{33}$$

This relation is different from the Einstein one already for low V and for $V=0$ coincides with the result obtained in Refs. [12,17] as the limiting case ($Q=0$) of the Fokker-Planck equation for dusty plasmas in the case of the dominant absorption collisions. Here we found this relation on basis of the theory with velocity-dependent coefficients, based on the probability transition approach developed above. For a more general form of the probability transition, which contains the free functions $\psi(P),\chi(q)$ and a small parameter $\zeta\ll 1$, $w(\mathbf{P},\mathbf{q})=\psi(P)\phi(|\mathbf{q}+\zeta\mathbf{P}|)\chi(q)$, it is possible to show that as above (with accuracy to $\sim m/M$) $\beta^*(V)\simeq\beta(V)$ and the relation between $\beta(V)$ and $D_\parallel(V)$ also has a form independent of V and different from the Einstein one:

$$\frac{\beta(V)}{D_\parallel(V)}=\frac{2\zeta M^2\int d\mathbf{q}q\chi(q)\left(\frac{\partial\phi(q)}{\partial q}\right)}{\int d\mathbf{q}q^2\chi(q)\phi(q)}. \tag{34}$$

This consideration shows that the structure of the Fokker-Planck equation for the processes, based on the Boltzmann-type collision integrals, is very different from the processes of other type when the Boltzmann-type collisions are not relevant. For the first type of processes the Einstein relation is valid, even for the case of velocity-dependent friction and diffusion coefficients, but in the limit of low grain velocity $|\mathbf{V}|\ll v_T$. For non-Boltzmann type momentum transferring, the fluctuation-dissipation theorem does not exist even for $V=0$, though some relation between the friction and diffusion coefficients can exist (specific for the each type of PT). These results have deep consequences for many physical systems, as well as for the systems of biological nature, e.g., cells moving in solutions and other, so-called, active walkers.

V. FRICTION AND DRAG FORCE IN DUSTY PLASMAS

In this section as an application of the developed theory, we consider the problem of momentum transfer from the ion stream to grains in dusty plasmas. Due to ion absorption and scattering by grains the drag force, acting on grains, appears. This force plays a crucial role in many experimental observations and, probably, is important for formation of voids in dusty plasmas for both "earthly" [18–20] and microgravity [21] conditions; the theory of voids essentially based on the drag force was given in Refs. [22,23]. The ion drag force $\mathbf{D}_I$ consists of two parts, so-called collection $\mathbf{D}_{Ic}$ and scattering $\mathbf{D}_{Is}$ ones. In the paper of Barnes *et al.* [24], the approximate analytical expressions for the drag collection and scattering forces were done. Later on the theory of drag force was actively developed analytically and numerically [23–28] to improve the description of drag force. It was achieved for the scattering part $\mathbf{D}_{Is}$, in particular, in the recent publications [27,28].

Here we are focusing on generalized description of friction and relation between the friction and drag forces. Let us calculate the generalized expression for the friction force $\mathbf{F}_f$ on the basis of the Fokker-Planck equation for the charged grains [Eqs. (3) and (29)]. We calculate here $\mathbf{F}_f$ only for ions because we are interested in the ion part of friction. Generalization for many species of the light components (electrons, atoms) is simple. By integration of Eq. (2) on momentum we find the following for the time evolution of the average momentum $\mathbf{P}_g$:

$$\frac{\partial(n_g\mathbf{P}_{gi})}{\partial t}=-\int d\mathbf{P}\tilde{A}_i(\mathbf{P},\mathbf{y})f_g(\mathbf{P},t). \qquad (35)$$

The function $\tilde{A}_i(\mathbf{P},\mathbf{y})$ is a generalization of $A_i(\mathbf{P})$ of Eq. (5) for the case of existence of some additional vector in the probability transition. In the case under consideration this vector $\mathbf{y}$ is the momentum, determined by the velocity of the ion stream $\mathbf{y}=M\mathbf{u}$:

$$\tilde{A}_i(\mathbf{P},\mathbf{y})=\int d\mathbf{q}q_i\tilde{w}(\mathbf{P},\mathbf{y},\mathbf{q}). \qquad (36)$$

To find $\tilde{w}(\mathbf{P},\mathbf{y},\mathbf{q})$, when there is ion flow in plasma, it is quite natural to use the shifted Maxwellian distribution function of ions.

At first we consider the scattering part $\mathbf{F}_{fs}$ of $\mathbf{F}_f$. The probability transition $\tilde{w}_s(\mathbf{P},\mathbf{y},\mathbf{q})$ in this case has the evident property

$$\tilde{w}_s(\mathbf{P},\mathbf{y},\mathbf{q})=w_s(\mathbf{P}-\mathbf{y},\mathbf{q}). \qquad (37)$$

Then we obtain the general expression for the friction force $\mathbf{F}_{fs}$:

$$\mathbf{F}_{fs}=-\int d\mathbf{P}(\mathbf{P}-\mathbf{y})\beta(|\mathbf{P}-\mathbf{y}|)f_g(P,t), \qquad (38)$$

$$\beta(\mathbf{P}-\mathbf{y})=\frac{1}{(\mathbf{P}-\mathbf{y})^2}\int d\mathbf{q}[\mathbf{q}\cdot(\mathbf{P}-\mathbf{y})]w_s(\mathbf{P}-\mathbf{y},\mathbf{q}), \qquad (39)$$

$$\tilde{\mathbf{A}}(\mathbf{P}-\mathbf{y})=(\mathbf{P}-\mathbf{y})\beta(|\mathbf{P}-\mathbf{y}|)=\int d\mathbf{q}\mathbf{q}w_s(\mathbf{P}-\mathbf{y},\mathbf{q}). \qquad (40)$$

In the limit cases of the scattering friction itself $\mathbf{F}_{f0s}\equiv\mathbf{F}_{fs}^{>}$ and scattering ion drag itself $\mathbf{D}_{Is}\equiv\mathbf{F}_{fs}^{<}$, Eq. (38) can be rewritten as

$$\mathbf{F}_{fs}^{>}=-\int d\mathbf{P}\mathbf{P}\beta(|\mathbf{P}|)f_g(P,t), \quad \mathbf{P}\gg\mathbf{y}, \qquad (41)$$

$$\mathbf{F}_{fs}^{<}=n_g\mathbf{y}\beta(|\mathbf{y}|), \qquad \mathbf{P}\ll\mathbf{y}. \qquad (42)$$

For the momentum distribution function of grains $f_g=\delta(\mathbf{P}-\mathbf{P}_0)$ (if the force is calculated for one particle) Eq. (38) takes the form

$$\mathbf{F}_{fs}=-(\mathbf{P}_0-\mathbf{y})\beta(|\mathbf{P}_0-\mathbf{y}|), \qquad (43)$$

and describes both the friction force itself and the drag force. In general, according to Eqs. (38) and (43), there is competition between friction and acceleration. For the limiting case of the friction force itself $\mathbf{F}_{f0s}$ for the grain with momentum $\mathbf{G}$ and immobile ions and the opposite case, ion drag itself $\mathbf{D}_{Is}$ with the ion velocity $\mathbf{u}=\mathbf{G}/\mathbf{M}$, there is a natural relation:

$$\mathbf{F}_{f0s}\equiv\mathbf{F}_{fs}^{>}(\mathbf{G})=-\mathbf{F}_{fs}^{<}(\mathbf{G})\equiv\mathbf{D}_{Is}(\mathbf{G}). \qquad (44)$$

This picture can be easily generalized for a few species of the light particles. From Eqs. (17) and (43) we find for $\mathbf{F}_{fs}$ the representation

$$\mathbf{F}_{fs}=\frac{m(\mathbf{V}-\mathbf{u})}{(\mathbf{V}-\mathbf{u})^2}\int d\mathbf{v}f_i(\mathbf{v}-\mathbf{V}+\mathbf{u})[(\mathbf{u}-\mathbf{V})\cdot\mathbf{v}]|\mathbf{v}|\sigma_{tr}(\mathbf{v}), \qquad (45)$$

$$\sigma_{tr}(\mathbf{v})=\int_{\chi_{min}}^{\chi_{max}}d\mathbf{\Omega}\frac{d\sigma}{d\mathbf{\Omega}}(1-\cos\chi). \qquad (46)$$

Here we use the limits for the angles of integration, taking account of providing convergence for the Coulomb cross section in the case of ion-grain scattering. Equation (45) coincides in the limit case $\mathbf{V}=\mathbf{0}$ with the well known general formulas for the transferring of momentum from light to heavy particle in the process of scattering, which can be justified from the simple physical arguments, as it was done, e.g., in Ref. [29]. For the opposite limit case $\mathbf{u}=\mathbf{0}$ it describes the friction force $\mathbf{F}_{f0s}$ for grain. This equation is also applicable, naturally, for the short-range scattering potentials, when $\chi_{min}=0$ and $\chi_{max}=\pi$. The specific result for $\mathbf{D}_{Is}$ in the case $|\mathbf{u}|\ll V_{T_i}$ can be written for the Coulomb cross section in the form

$$\mathbf{D}_{Is}=2A_0M\mathbf{u}\Gamma^2\ln\Lambda, \qquad (47)$$

where $A_0=(\sqrt{2\pi}/3)(m_i/M)a^2n_iv_{T_i}$ and $v_{T_i}\equiv\sqrt{T_i/m_i}$. The parameter $\Gamma\equiv e^2Z_gZ_i/aT_i$ and usually $\Gamma\gg1$ for dusty plasmas. The structure of the generalized Landau logarithm $\ln\Lambda$ for dusty plasmas is very important and has been recently considered in Refs. [27,28]. For very strong interaction, the problem of the correct form of $\ln\Lambda$ as function of plasma parameters is still not completely solved.

Let us consider now the generalized collecting friction $\mathbf{F}_{fc}$ and in particular the collecting drag force $\mathbf{D}_{Ic}$. The formula of structure similar to Eq. (45) but with $\mathbf{V}=\mathbf{0}$ and with the collecting nontransport cross section, instead of transport scattering cross section, was applied also for the collecting drag force in Refs. [23,27,28] and other papers on phenomenological basis. Our goal here is to investigate the collecting drag force by use the PT Eq. (32) for absorption and to find the relation between the friction force and the friction coefficient for the collecting process. Recently in Ref. [13] the friction coefficient $\beta(V)$ in dusty plasmas was calculated explicitly for arbitrary grain velocity and parameter Γ. It was found that $\beta(V)$ can change sign (''negative friction'') from positive to negative for some velocity domain if the parameter $\Gamma>1$. Here we reproduce the result of Ref. [13] for the total friction coefficient for a grain $\beta(V)$ [for the particular but important case $\eta\equiv(m_i v^2/2T_i)\ll 1$ and arbitrary Γ]:

$$\beta(\eta,\Gamma)=2A_0\left[1-\Gamma+4\frac{n_a}{n_i}\left(\frac{T_a m_a}{T_i m_i}\right)^{1/2}-\frac{\eta}{5}(1-3\Gamma)+\Gamma^2\ln\Lambda\right]. \quad (48)$$

The terms in Eq. (48) proportional to the atom density n_a and to $\ln\Lambda$ describe friction, respectively, with atoms and with ions by scattering. These terms are always positive. Other terms in Eq. (48) describe negative friction due to ion absorption by grains and are negative in the considered limit case $\eta\ll 1$ if $\Gamma>1$. Negative friction exists for small η if the Coulomb scattering is strongly suppressed, when the Coulomb logarithm $\ln\Lambda$ is small [13,27] (some additional reasons for its reduction are discussed in Ref. [28]), which is typical in strong interaction in dusty plasmas. The level of ionization has to be high enough to provide negative value of the friction coefficient. As we know, these conditions in present are not reached in the experimental setups. Opportunity for manifestation of negative friction in the experiments requires as we already mentioned the special conditions. From Eqs. (35), (36), and (48), it follows straightforward by that the collecting friction force itself $\mathbf{F}_{f0c}$ for a moving grain can be written as

$$\mathbf{F}_{f0c}=-\mathbf{P}_0\beta(\eta,\Gamma). \quad (49)$$

To find the generalized collecting friction force $\mathbf{F}_{fc}$ (and therefore also the drag force), we have found the function $\tilde{w}_c(\mathbf{P},\mathbf{y},\mathbf{q})$ for collection. The crucial fact is that the relation similar to Eq. (37) or (49) for PT function is not correct for the absorption in the case when an ion flow exists ($\mathbf{u}\neq\mathbf{0}$), due to the different structure of the PT functions for the scattering and collection processes. It is the property of the existing kinetic models for absorption in dusty plasmas, in which the surface recombination and atom emission is not taken into account explicity.

To describe the ion stream with the velocity $\mathbf{u}\neq\mathbf{0}$ we use again the shifted Maxwellian distribution of ions. As it is easy to see in this case the PT function $\tilde{w}_c(\mathbf{P},\mathbf{y},\mathbf{q})$ can be expressed via w_c determined by Eq. (32):

$$\tilde{w}_c(\mathbf{P},\mathbf{y},\mathbf{q})=w_c(\mathbf{P}-\mathbf{y},\mathbf{q}+m\mathbf{u}). \quad (50)$$

For the momentum distribution function of grains $f_g=\delta(\mathbf{P}-\mathbf{P}_0)$ (if the force is, as above, calculated for one particle), the friction $\mathbf{F}_{fc}$ takes the form

$$\mathbf{F}_{fc}=-\tilde{\mathbf{A}}(\mathbf{P}_0,\mathbf{y})=-\int d\mathbf{q}(\mathbf{q}-m\mathbf{u})w_c(\mathbf{P}_0-\mathbf{y},\mathbf{q}). \quad (51)$$

This equality can be written in the equivalent form

$$\mathbf{F}_{fc}=m\mathbf{u}\beta_1(|\mathbf{P}_0-\mathbf{y}|)-(\mathbf{P}_0-m\mathbf{u})\beta_2(|\mathbf{P}_0-\mathbf{y}|), \quad (52)$$

where the coefficients β_i are related with the zero and first moments of the PT function:

$$\beta_1=\int d\mathbf{q}w_c(\mathbf{P}_0-\mathbf{y},\mathbf{q}), \quad (53)$$

$$\beta_2=\frac{1}{(\mathbf{P}_0-\mathbf{y})^2}\int d\mathbf{q}[\mathbf{q}\cdot(\mathbf{P}_0-\mathbf{y})]w_c(\mathbf{P}_0-\mathbf{y},\mathbf{q}). \quad (54)$$

Let us consider the simple and practically important case, when both vectors $\mathbf{P}_0=Mv_0$ and $\mathbf{y}$ are directed parallel or antiparallel to the same unit vector $\mathbf{l}$: $\mathbf{P}_0=\pi_0\mathbf{l}$ and $\mathbf{y}=y_0\mathbf{l}$. Then for the friction, we arrive at the following expression:

$$\mathbf{F}_{fc}=\mathbf{l}\{u_0[m\beta_1(|\pi_0-y_0|)+M\beta_2(|\pi_0-y_0|)]-\pi_0\beta_2(|\pi_0-y_0|)\}. \quad (55)$$

Here and below we use velocities related with the momenta $\pi_0\equiv Mv_0$ and $y_0\equiv Mu_0$. Equation (55) can be represented in the equivalent and explicit form:

$$\mathbf{F}_{fc}=-\mathbf{l}\left\{mu_0\int d\mathbf{v}\frac{\mathbf{v}\cdot\mathbf{l}(u_0-vv_0)}{(u_0-v_0)^2}f_i[\mathbf{v}+\mathbf{l}(u_0-v_0)]|v|\sigma_c(|v|)-mv_0\int d\mathbf{v}\frac{[\mathbf{v}\cdot\mathbf{l}+u_0-v_0](u_0-v_0)}{(u_0-v_0)^2}\times f_i[\mathbf{v}+\mathbf{l}(u_0-v_0)]|v|\sigma_c(|v|)\right\}. \quad (56)$$

Let us consider the special cases of Eq. (56).

(a) $v_0\gg u_0, u_0\to 0$. In this case we arrive at the friction force $\mathbf{F}_{f0c}$:

$$\mathbf{F}_{f0c}=-m\mathbf{V}_0\int d\mathbf{v}\frac{[\mathbf{V}_0\cdot(\mathbf{v}-\mathbf{V}_0)]}{V_0^2}f_i[\mathbf{v}-\mathbf{V}_0]|v|\sigma_c(|v|). \quad (57)$$

This expression coincides with the collecting ion friction force, which leads to the negative collecting friction coefficient for $\Gamma>1$ [13] and to the respective relations (48) and (49).

(b) $u_0\gg v_0, v_0\to 0$. In this case the generalized collecting friction $\mathbf{F}_{fc}$ describes the collecting ion drag force $\mathbf{D}_{Ic}$:

$$\mathbf{D}_{Ic}=m\mathbf{u}\int d\mathbf{v}\frac{\mathbf{u}\cdot\mathbf{v}}{u^2}f_i[\mathbf{v}-\mathbf{u}]|v|\sigma_c(|v|). \quad (58)$$

This equation coincides with the expression for the collecting drag force, which has been suggested in Refs. [27,28].

(c) $u_0 \neq 0$, $v_0 \neq 0$. Temperature of ions is low $|(u_0 - V_0)| \gg v_{Ti}$, the ion distribution function tends to the δ function $n_i \delta(\mathbf{v}+\mathbf{u}-\mathbf{V}_0)$. Then $\mathbf{F}_{fc}$ tends to the force directed along the ion stream, which we denote as $\mathbf{D}_{I0c}$:

$$\mathbf{D}_{I0c} = m n_i \mathbf{u} |(u_0 - v_0)| \sigma_c(|(u_0 - v_0)|). \tag{59}$$

As it follows from Eq. (59) ion wind in the considered limit always accelerates grains. If u_0 is parallel to V_0 and they are close one to another, but both are higher than v_{Ti}, the enhancement of the drag force $\mathbf{D}_{I0c}$ occurs with decrease of the relative velocity $|(u_0 - V_0)|$. It is the consequence of the OML collecting cross section, which probably can be observable for very fast grains, discovered in some experiments [30,31], or in cryogenic discharges.

VI. ACTIVE PARTICLES

During the last decade, investigation of motion of the self-organized objects, e.g., cells, is in a focus of interest, due to numerous measurements and observations of their dynamical behavior [8,32]. Our goal in this work is to show that construction of the relevant probability transition on basis of the simple physical requirements permits to justify the relevant description of such systems. In particular, we show that for the simplest structure of PT for motion of an active particle, the known (and experimentally verified) structures of the velocity distribution of grains (cells) [32], which are able to have a directed motion near a fixed nonzero velocity, can be justified. The coefficients of this distribution are calculated.

Let us formulate some general conditions to find the structure of the PT for active particles. We suppose that the linear momentum, transferred from a grain (cell) to the surrounding medium, is created by loss of the inner energy of this grain. Below we ignore the processes of energy supply, which can be included separately in more complicated schemes.

At first we assume that the transferred to medium momentum $\mathbf{q} \ll \mathbf{P}$ is distributed near some fixed value $\mathbf{q}_0$. The frequency of generation of the transferring momentum $\mathbf{q}$ will be denoted as $\mu(\mathbf{q})$. It can be approximated, for example, by the product of the functions, describing distributions on modulus $|\mathbf{q}| = q$ and on the space angles θ, φ between the vectors $\mathbf{q}$ and $\mathbf{P}$, if there are no other, besides $\mathbf{q}$ and $\mathbf{P}$, characteristic vectors for the system. In this simplest case we can put $\mu(\mathbf{q}) = \nu_0 \mu_1(q) \mu_2(\theta, \varphi)$, where ν_0 is the q-independent frequency of momentum generation by a grain (cell). Below we suggest that the φ dependence of μ_2 is absent. The distribution $\mu_1(q)$ can be Gaussian or, for the limit case of very narrow distribution of the transferring momentum, can be approximated by the δ function $\mu_1(q) = \lambda_0 \delta(q - q_0)/ q_0^2$, where λ_0 is a dimensionless constant. For the function $\mu_2(\theta)$ we assume, that the angle θ between the direction of the transferred momentum and the momentum of the active grain is enclosed between the values $\pi - \theta_0 < \theta < \pi$, where θ_0 is some acute angle. Due to this, amplification of the grains takes place. Consideration for the 2D and 1D cases is evident.

The "weight function" $\Delta(\theta)$ for the angle θ can be included to describe the axis-symmetrical nonhomogeneity of amplification for the different angles θ. We can also include two additional weight functions: $\Sigma(P)$ and $\Upsilon(\varepsilon - \varepsilon_0)$. The first one describes dependence of PT from the modulus P of the momentum $\mathbf{P}$, the second one describes that momentum transferring is possible only if the inner energy of a grain (cell) ε is bigger than some fixed minimal value of the inner energy ε_0, let us say $\Upsilon(\varepsilon - \varepsilon_0) = \vartheta(\epsilon - \varepsilon_0)$, where ϑ is the steplike function.

Under these assumptions, the PT for the momentum transferring w_ε, due to loss of the inner energy of an active particle, can be written in the case under consideration as

$$w_\varepsilon(\mathbf{P}, \mathbf{q}) = \nu_0 \Upsilon(\varepsilon - \varepsilon_0) \Sigma(P) \Delta(\theta) \mu_1(q) \vartheta(\theta - \pi + \theta_0) \times \vartheta(\pi - \theta). \tag{60}$$

The values of the friction and diffusion coefficients follow from general Eqs. (8)–(11):

$$\beta_\varepsilon(V) = -2\pi \nu_0 \Upsilon(\varepsilon - \varepsilon_0) \frac{\Sigma(P)}{P} \int_0^\infty dq q^3 \mu_1(q) \times \int_{\pi - \theta_0}^{\pi} d\theta \sin\theta \cos\theta \Delta(\theta), \tag{61}$$

$$D_\varepsilon(V) = 2\pi \nu_0 \Upsilon(\varepsilon - \varepsilon_0) \frac{\Sigma(P)}{M^2} \int_0^\infty dq q^4 \mu_1(q) \times \int_{\pi - \theta_0}^{\pi} d\theta \sin\theta \cos^2\theta \Delta(\theta). \tag{62}$$

To find the total expressions for $\beta(P)$ and $D(p)$ we have added to the values (61), (62) of $\beta_\varepsilon(P)$ and $D_\varepsilon(P)$ the parts of the friction and diffusion coefficients aroused due to collisions between the cell, moving with the momentum P, and the surrounding particles (atoms) of the solution. For the case of 3D elastic collisions and the hard sphere interaction, these parts were calculated on the basis of Eqs. (21) and (22). For the velocities of cells essentially less than the characteristic velocity of atoms we can ignore the velocity-dependent multipliers and consider the parts of the coefficients $\beta_{el}(V) = \beta_0$ and $D_{el}(V) = D_0$, connected with the elastic collisions as constants. These constants as well as the initial velocity-dependent functions (for $V < v_{Ta}$) [see Eq. (23)] are connected by the Einstein relation $D_{el}(V) = M^{-1} T \beta_{el}(V) \simeq M^{-1} T \beta_0$.

If we make the natural assumption that $\Sigma(P)$ is a constant, which means that the PT w_ε is not dependent on the cell velocity (as the process, which is determined by the inner state of the grains), we find from Eq. (61) that $\beta_\varepsilon(P) \sim 1/P$. Due to this specific dependence, the Fokker-Planck equation for cells can be written as

$$\frac{df_g(V)}{dt}=\frac{\partial}{\partial V}\left\{[V\beta_0-K_\varepsilon]f_g(V)+[D_0+D_\varepsilon]\frac{\partial f_g(V)}{\partial V}\right\}, \tag{63}$$

where $K_\varepsilon>0$ is a constant, determined by the equality $\beta_\varepsilon \equiv -K_\varepsilon/V$ and Eq. (61), and D_ε is determined by Eq. (62). For β_0 Eq. (21) can be used. Finally, the result is $\beta_0 = 8\sqrt{2Tmn_a}S/(3M\sqrt{\pi m})$, where $S=\pi a^2$ is the area of the grain and n_a is the density of the atoms. The stationary solution of Eq. (63) is the Gaussian distribution:

$$f_g(V)=C\exp\left\{-\frac{\beta_0}{2D_\Sigma}\left(V-\frac{K_\varepsilon}{\beta_0}\right)^2\right\}. \tag{64}$$

Here C is the constant of normalization and $D_\Sigma\equiv(D_0+D_\varepsilon)$.

The velocity dependence of the distribution function $f_g(V)$ (64) coincides with the one that has been found in Ref. [32] on the basis of the phenomenological assumption, concerning the structure of the friction coefficient in the Langevin equation. This type of the velocity distribution function in our consideration is the consequence of physically clear choice of the perturbation transition function. It can be generalized for more complicated and practically important cases, when there are one or more (additional to **P**) vectors, which determine the direction of **q**. It can be some inner vector, ''driver,'' which can be orientated on the external (e.g., surface) gradients of density, or temperature, or concentration of some ingredient in the ambient medium. In that case, naturally, the equilibrium state is not effectively one dimensional. Active particle (e.g., cell) can turn during motion. These problems will be considered separately.

VII. CONCLUSIONS

Here we use the simple and effective way for concretization of the Fokker-Planck equation on basis of self-consistent determination of the friction and diffusion coefficients. Both are determined as the functionals of probability transition. This function possesses a very different structure for the Boltzmann-type collisions and the other ones. We found PT for the Boltzmann-type collisions and proved that velocity-dependent friction and diffusion coefficients are connected by the Einstein relation for the velocities of grain less than thermal velocity of the small particles. At the same time, there is a crucial violation of the Einstein relation for the higher grain velocity. Therefore, in general, for the velocity-dependable friction and diffusion coefficients even for the Boltzmann-type collisions the applicability of the Einstein relation is limited by not very high (nevertheless practically most important) values of the grain velocity. The velocity dependence of these coefficients and renormalization of the friction coefficient in 2D and 3D cases as a consequence of the tensorial structure of diffusion are found. Because the Fokker-Planck equation is single valued also for the velocity-dependent coefficients, the problem of connection between Langevin and Fokker-Planck equation has to be reformulated as a problem of the relevant (to the Fokker-Planck equation) Langevin equation.

For the non-Boltzmann collisions, e.g., for the absorption collisions, the structure of PT follows from the structure of the collision integral, obtained earlier [11,14,15] and leads, in particular, to the relations different from the Einstein's relations between the coefficients in the relevant Fokker-Planck equation already for the region of low grain velocities.

As the example of application of the PT method to the more complicated systems we considered the generalized friction force in dusty plasmas. The scattering and collecting parts of this force are determined by the generalized friction coefficient, as a function of ion stream and grain velocities. The remarkable fact is that the sign of the collecting friction coefficient can be negative for some plasma parameters, as it was recently shown [13]. Of course realization of negative total friction coefficient for grains for dusty plasmas in experiment requires special conditions, because other mechanisms of friction exist. The ion drag force in such approach as well as the friction force itself are the particular cases of this generalized friction. The ion scattering and collecting drag forces are found and calculated for the various particular cases. Some phenomenological expressions, which have been used for calculations before, are rigorously proved and generalized. To compare theoretical results with the experiments in dusty plasmas, a more detailed description of forces, which takes into account, in particular, the process of surface recombination and mass transfer by atom evaporation has to considered. Such type kinetic theory is in the stage of development and will be published separately.

We also constructed the PT for the active particles (e.g., grains or cells) in an ambient medium for some simple situation. On basis of physically clear assumptions we found that the part of the generalized friction coefficient, responsible for self-motion, can possess the peculiarity $1/P$, where P is the momentum of a grain. Given also the appropriate usual friction mechanism, the stationary solution of the relevant Fokker-Planck equation is Gaussian with a peak around some nonzero velocity. Some generalizations of the obtained results for more complicated cases are suggested.

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